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DESIGN AND ANALYSIS OF NON-UNIFORM RATE DIGITAL CONTROLLERS

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Abstract: Various applications in digital systems require the involvement of concepts from signal processing and filtering. These specific problems often need the linear dynamic systems to have a transfer function that can specify the behavioural characteristics of the system. When operating in the digital domain, such functions can effectively be used to approximate the same characteristics over the frequency range of importance as any given continuous-time transfer function. In the case with uniform sampling, linear systems theory can directly provide an answer to determine the frequency response. However, when the element of randomness is added to the sample rate of the discrete controller, the common analysis technique of substituting $z{=}e^{jw}$ will not give the correct result. This paper therefore places an emphasis on the Fourier analysis and highlights a technique to compute the magnitude and phase of a non-uniform rate transfer function at various frequencies in the time domain.

Keywords: non-uniform sampling, Fourier analysis, delta operator, digital control

1. INTRODUCTION

Non-uniform sampling has shifted into an era of research where its theoretical analysis can be realized as a practical solution. Whilst traditional engineering fields have always been aimed towards uniform sampling, irregular sampling is slowly becoming the focal point for research as an ultimate cheap alternative for countering issues that otherwise cannot be solved when using uniform sample rates. In signal processing applications, the methodology of non-uniform sampling has enabled the processing of digital signals at much slower rates without restrictions from the well-known Nyquist limit (Artyukh et al. [1997], Papenfuss et al. [2003]). Carefully designed sampling schemes [1] can therefore be used to effectively mitigate the effects of aliasing and permit significant reductions in the average sampling frequency, leading to more efficient processor utilisation. Summarising the application potential with non-uniform sampling, it has been used for the:

- reduction in the bit flow process (Artyukh et al. [1997])
- use of simpler electronics due to a reduction in overall processing (Bilinskis [2007])
- simplification of complex designs (Sonnaillon and Bonettot [2007])

In classical digital controller design, the sampled signals are always considered to be periodic. However, in practise some sample time variations are inevitable during operation. Furthermore, in industrial environments it is usually very difficult to preserve the simplification of a constant sample rate, and much effort has been researched to reduce

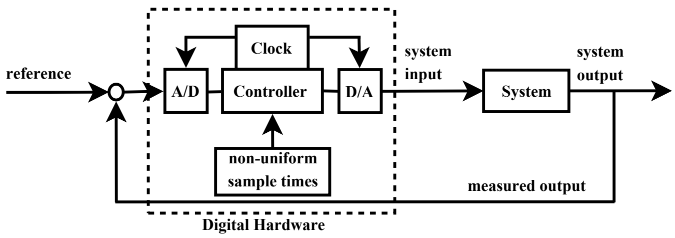

Fig. 1. Non-uniform rate discrete equivalent control system

such effects (Albertos and Crespo [1999]). More recently, Bilinskis [2007] coined together the term *'alias-free signal processing'* which demonstrates the reality of expanding the frequency range of a signal without being corrupted by aliasing. The implications of the phrase *alias-free* are that it is possible to determine information at frequencies well in excess of the average sample rate. The corollary for control, perhaps, is that the effect of sampling delays could be reduced such that the average sampling frequency does not need to as high(Goodall [2001]).

Consequently, the motivation for this research has been to investigate practical ways of creatively using these variations in the sample period or even utilizing a deliberate non-uniform sampling scheme to enabling a lower sampling frequency, without compromising the operating bandwidth of the digital compensator. In addition, such models can provide the basis for very valuable testing for control systems with varying sampling periods prior to any practical implementation, where situations with *jitter* sampling can be analysed (Marti [2002]).

Fig. 1 illustrates a typical non-uniform control setup is composed of the usual elements — the inputs/outputs

[1] —that are independent of the input signal.

signals, with the outputs being generated by passing the inputs through the analog plant and sensor dynamics. Hold devices (such as digital-to-analog converters) are then used to maintain a continuous signal driving the plant. The only addition to the setup is a sampler block which regulates the sample times and reconfigures the discrete filter. The complete design and analysis of such a control system is lacking in many ways since most of the tools and techniques, cited in control literature, are exclusively based on either continuous-time or uniform sampling discrete-time techniques. A common technique used for closed-loop analysis is to determine the steady-state frequency response of the system, often plotted using the Bode plot. It represents the natural behaviour of a linear system over a range of selected frequencies. With uniform sampling, linear systems theory can directly provide the answer to the frequency characteristics i.e. simply by substituting $z = e^{jw}$ in the z transfer function. Unfortunately, the same may not be used to perform an accurate frequency analysis for the case with non-uniform sampling and, therefore, this paper proposes the use of the Fourier analysis to provide an appropriate result.

This paper is structured as follows: Section 2 summarise the requirements of sampling in digital control. Section 3 presents the non-uniform rate discrete equivalent controller designed using the modified canonic δ transform, which was developed in Khan et al. [2008]. Section 4 demonstrates a frequency response estimation technique based on the Fourier analysis. It is a combination of frequency domain and time domain analysis that can effectively be used to calculate the frequency response under uniform and non-uniform sampling conditions. The rest of the sections present an example implementing the technique and finally draws a conclusion to the paper.

2. UNIFORM SAMPLING REQUIREMENTS IN DIGITAL CONTROL

The selection of the sampling frequency of a digital control system is usually a compromise among many aspects of the design. The basic motivation for lowering the sample rate is cost (Clarke and Maslen [2007]). A slow sample rate directly reduces the hardware costs and makes its possible for a slower computer to achieve a given control function; or provides greater capability for a given computer. The potential disadvantages of slow sampling, relative to controller bandwidth, may lead to open loops between samples [2] or a control input with large steps [3]. On the other hand, a very fast sample time can assure stability and performance of a system, based on certain selection criterion that can provide the overall frequency response after the reconstruction process.

It should be noted that the single most important impact of the sample rate in a control system is the delay associated with the reconstruction device (ZOH), which degrades the degree of system stability. The phase delay introduced is approximately:

$$\phi_s = 360 f_0 \frac{1}{2f_s}, \qquad f_0 \le f_s \tag{1}$$

where ϕ_s is the phase delay in degrees introduced by sampling, f_s is the sampling frequency and f_0 is the signal bandwidth. Wu [2005] argues that to achieve a phase delay no more than $5\,^\circ$, the corresponding sampling frequency should be increased by 72 times the operating bandwidth of the control system. This criterion should be enough to suffice the control stability requirements and this high sample frequency is why particular emphasis has to be placed on the numerical requirements for the controller.

In digital control, usually a relatively high sample rate is used; for example, a figure of around 100 times the control system bandwidth is often recommended (Goodall et al. [1998]). However, there might be certain cases in which the processing system cannot meet this requirement, indicating that a higher performance device is required (along with a rise in the hardware cost). Therefore the authors are exploring the option of using a non-uniform sampling scheme to reduce the average sample rate, leading to a reduction in the controller processing and hardware requirements.

3. NON-UNIFORM SAMPLING

Several sampling schemes have been investigated in the area of digital signal processing with distinct properties (Bilinskis and Mikelsons [1990]). One of the most promising is the additive random sampling scheme, which can remove aliasing without any preprocessing requirements, thereby reducing the complexity of the system. This allows high-frequency analogue signals to be sampled at much lower sample rates and yet avoid the addition of any aliases in their digital spectra. More recently, other texts have commented on non-uniform sampling theory and its applications (see e.g. Marvasti [2001], Bilinskis [2007]), demonstrating its advantages and benefits. However, despite being a popular area for research in digital signal processing, it has received scant attention in the field of digital control. For the most part, control theory disregards non-uniform sampling in the controller design of continuous-time linear systems (Marti et al. [2001a,b]). This could be due to the lack of convenient analysis techniques and the fact that unintended variations in the sampling instants could cause degradations in the control performance *or* may even lead to instability (Marti [2002]). Either way, to the authors' knowledge, there has been no research reported investigating the opportunities of using an intentional non-uniform sample rate in feedback control systems and therefore the opportunity for research in this area is unique.

3.1 Non-uniform rate discrete equivalents

Controller design is usually done in the continuous plane and later, once a sampling frequency is selected, the discrete-time equivalent is computed and used to replace the continuous-time design. The problem when dealing with a non-uniform sample rate poses an obvious question: *Can the z-transform equivalent be used in a time varying sampling frequency?* The answer of this question was addressed in Khan et al. [2008], where the authors made

[2] The sampled output will be a poor representation of the actual continuous-time response; inter-sample ripple (Goodwin and DeSouza [1984]).

[3] These can feed significant energies to high frequency mechanical resonances.

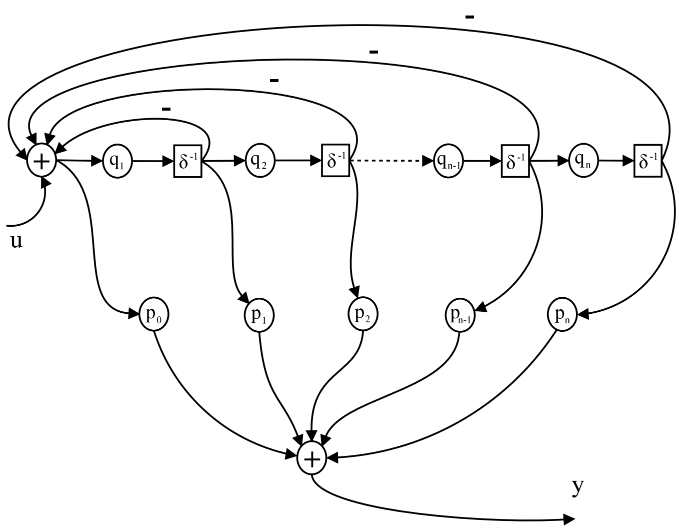

Fig. 2. The generalised modified canonic δ-filter

some simplifying assumptions in order to obtain a more tractable system model for design purposes.

The z-transform can provide a ready access to the system response characteristics and stability margins by determining the control gains and filter coefficients effectively be used for analysing discrete-time filters operating with a uniform sample frequency. However, due to the involvement of the random time variable in the non-uniform setup, the transform might not necessarily exist. Nonetheless, if the sample time instants are known before hand, the transfer function characteristics can be adjusted by recalculating its coefficients in the z-transform according to the changing sampling frequency. This approach is based on the same classic controller design methods used in control theory. It is also pointed out that it is essential to chose the proper implementation structure at the design stage (Kovacshazy et al. [2001], Khan et al. [2008]). This can significantly reduce the intolerable effects of digital filter reconfigurations during real-time implementations. Therefore, the designed non-uniform rate control algorithm, based on the modified canonic δ approach (Forsythe and Goodall [1991]) was proposed to cope with sample time variations. Diagrammatically the controller is illustrated in Fig. 2. The generalised transfer function with the modified δ canonic takes the form:

$$\frac{y}{u} = \frac{p_0 + p_1 q_1 \delta^{-1} + p_2 q_1 q_2 \delta^{-2} + \ldots + p_n q_1 q_2 \ldots q_n \delta^{-n}}{1 + q_1 \delta^{-1} + q_1 q_2 \delta^{-2} + \ldots + q_1 q_2 \ldots q_n \delta^{-n}} \tag{2}$$

Moreover, since the sampling periods are predefined, the control algorithm recalculates the controller coefficient values depending on the sample rate. The derivation of the *coefficient calculation equations*, to be used in Eq. 2, are derived in Khan [2010].

4. FREQUENCY ANALYSIS METHOD

In general, a Fourier analysis can be performed over a running window of the fundamental frequency, that can allow the calculations of the magnitude and phase of the observation signal. A much more comprehensive description of the analysis technique, its use, and its limitations is given in Katznelson [1976]. The easiest

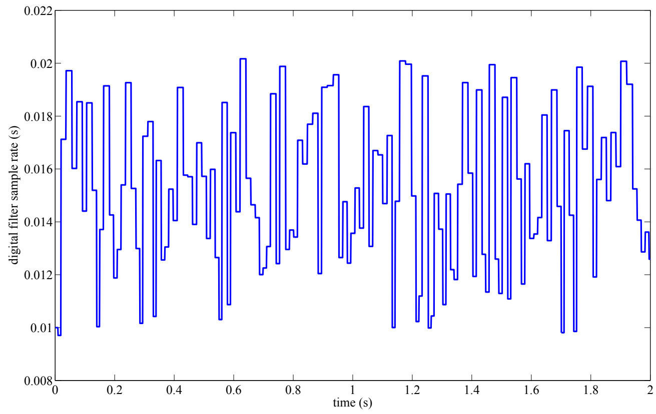

Fig. 3. The non-uniform sample rate (with uniform distribution), $T_n \sim$ U(0.01,0.02)

way to calculate the frequency response of a system is to substitute $s \rightarrow jw$ in the continuous-time transfer function and obtain the magnitude and phase at different frequencies. However, when the element of randomness is added to the sampling frequency of the discrete controller, an important question arises: *How to analyse the frequency response of a system with a non-uniform rate controller?*

It may be the case that the maximum and minimum sample rates, being used in the experiment, can be used to draw out the confidence regions on the Bode plot with uniform sampling. However, this technique is only an approximation and does not provide an accurate result during run-time. Speculating the concept for non-uniform sampling, the most probable domain to find a solution is to revisit the basic foundations of signal analysis i.e. the Fourier series. As widely known, the Fourier analysis is a process that decomposes a given signal into various sinusoids of different frequencies. These sinusoids are actually the harmonics of the fundamental frequency of the original function that is being analysed.

4.1 Evolution of the Fourier Coefficients

Rethinking the basic Fourier definition, a periodic signal $f(t)$ can be expressed by a Fourier series in the form:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(t)\cos(nw_1 t) + b_n(t)\sin(nw_1 t) \tag{3}$$

where $\frac{a_0}{2}$ is the DC component or *the average value of the signal*, and n represents the rank of the harmonics ($n = 1$, corresponds to the fundamental component). The remaining variables can be described as:

$$a_n(t) = \frac{2}{T_F} \int_{t-T_F}^{t} f(t)\cos(nw_1 t)dt$$

$$b_n(t) = \frac{2}{T_F} \int_{t-T_F}^{t} f(t)\sin(nw_1 t)dt$$

$$T_1 = \frac{1}{f_1} = \frac{2\pi}{w_1}$$

$$T_F = kT_1, \qquad k > 0$$

Frequency (Hz)	Continuous (dB)	Uniform rate (dB)	Non-uniform rate (dB)	
			max	min
5	12.11	11.88	12.26	11.48

Table 1. The filter magnitude values (dB)

Frequency (Hz)	Continuous (degrees)	Uniform rate (degrees)	Non-uniform rate (deg)	
			max	min
5	-25.65	-38.8	-36.5	-42

Table 2. The filter phase values (degrees)

where f_1 is the fundamental frequency and T_F is the integration time being averaged via a moving window over k periods of the fundamental for the Fourier analysis.

The definition of the Fourier coefficients, a_n and b_n, presented in Eq. 3 are considered to be functions of time and hence can be used to describe the behaviour of the signal frequency characteristics in the time domain. The magnitude and phase of the observation signal $f(t)$, or the *selected harmonic component*, can be calculated by the following equations:

$$\angle H_n = \arctan\left(\frac{a_n(t)}{b_n(t)}\right)$$
$$|H_n| = \sqrt{a_n^2(t) + b_n^2(t)} \tag{4}$$

After performing the Fourier analysis of the filter output and input signals, the frequency response of the transfer function is computed by:

$$Gain = \frac{H_{output}}{H_{input}}$$
$$Phase(\varphi) = \angle H_{output} - \angle H_{input} \tag{5}$$

The steady state response of a system can be evaluated for a sinusoidal input at a given frequency. For a continuous-time system, the response will be the same frequency as the input, but the frequency response parameters will be modified with respect to the transfer function of the system being assessed under the input frequency [4]. Example A demonstrates the method with a 0.5Hz input sinusoid signal (amplitude = 1). The frequency response parameters of the transfer function under observation can thus be obtained by performing a complete analysis with different frequency values. These can then be compared with the ideal frequency response to understand the characteristics when using a non-uniform rate setup.

Example A Consider a 2^{nd} order phase lag-lead compensator, the continuous transfer function of which can be given as:

$$H(s) = \frac{0.0025s^2 + 0.35s + 10}{0.0005s^2 + 0.105s + 1}$$

The non-uniform sample rate being used in this exercise has a uniformly distributed set of samples between 0.01s and 0.02s, as illustrated in Fig. 3. Therefore, the average non-uniform sample rate for the process is 0.015s. Using the approach highlighted in Section 3.1, a formal Fourier analysis is performed on the transfer function with a moving window of k=3. Figs. 4 and 5 illustrates the technique performed with the continuous filter, the uniform

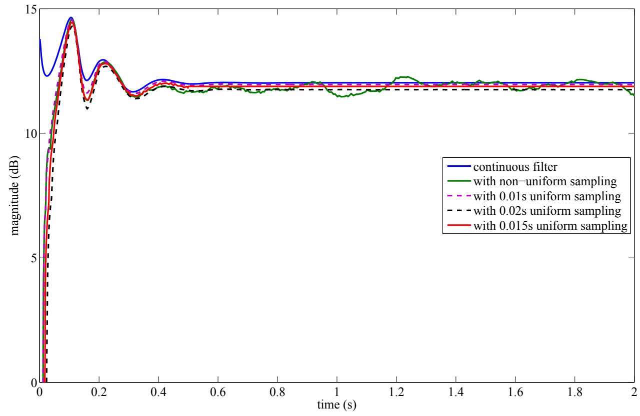

Fig. 4. The filter magnitude response at 5Hz

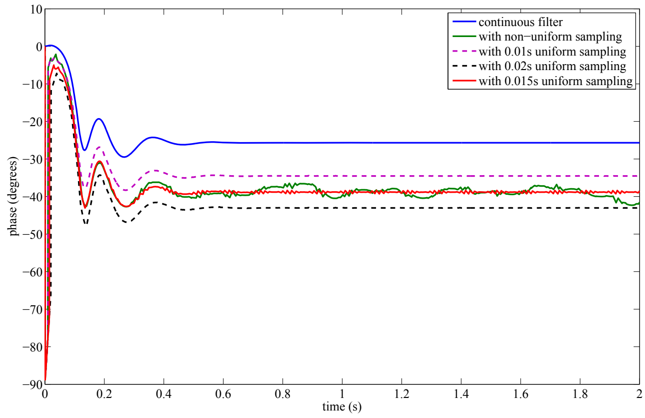

Fig. 5. The filter phase response at 5Hz

filter (with sample rate at 0.015s) and the non-uniform sampling filter (with uniformly distributed samples at an average sampling frequency of 0.015s). They depict the behaviour of the magnitude and phase at the fundamental frequency of 5Hz after reaching the steady state value [5]. The magnitude plot suffers from sudden gains due to filter reconfigurations or when the digital filter coefficients switch from one operational mode to another. These gains obviously depend on the amount of change in the sample rate. On the other hand, the phase plot is bounded and oscillates between the maximum (0.01s) and minimum (0.02s) sample rates being used in the experiment. Note that there is an increased phase lag even for the case with uniform sampling, which is the sampling delay mentioned earlier. The technique allows the process to observe in real-time and plots the 'Evolution of the Fourier Transform coefficient' calculations in the time domain. From the simulation, the frequency response parameters are tabulated in Tables 1 and 2.

The scope of the technique may further be extended to observe the changes that may occur in the magnitude and phase of non-uniform rate controllers over time. Such plots provide a much accurate answer to the frequency response characteristics with non-uniform sample rates.

[4] On the contrary, the response of a digital system to an input signal will consist of a sum of many sinusoids spaced at integer multiples of the sampling frequency.

[5] The Fourier result reaches to steady-state values after an initial computational transient. The later simulations carried out in this paper using this technique will present **only** the final steady-state values i.e. after the transient stage has died out.

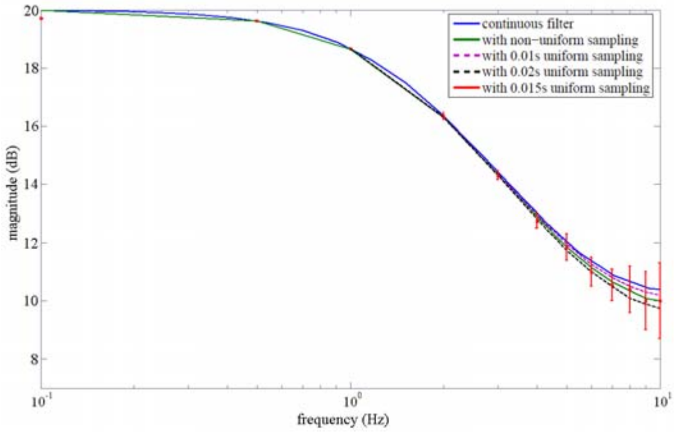

Fig. 6. Digital filter magnitude estimation

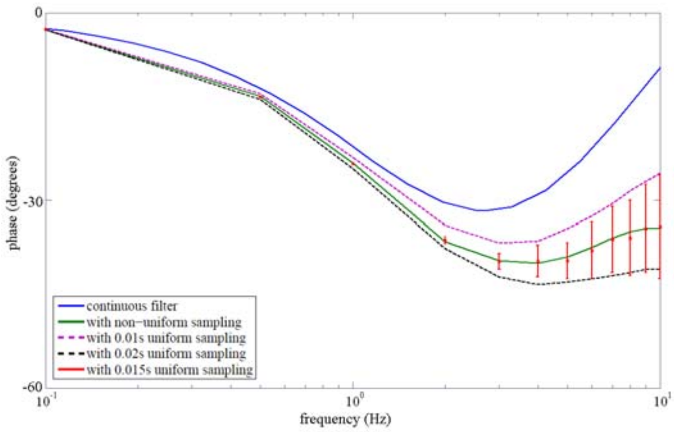

Fig. 7. Digital filter phase estimation

Furthermore, this procedure can be repeated for a range of frequencies and a complete Bode diagram can now readily be plotted for the non-uniform rate digital filter.

5. APPROXIMATE FREQUENCY DOMAIN RESPONSE

The method demonstrated using Example A can effectively be used to obtain the dynamic frequency response of a digital compensator. It is a combination of the frequency domain and time domain analyses based on the Fourier series. It can be used to evaluate the magnitude and phase characteristics of uniform and non-uniform sample rate systems.

Simulation setup

The analysis here consists of three compensators:

- the continuous controller
- the discrete controller with a uniform sampling rate T_n of 0.015s
- the discrete controller with non-uniform sampling with uniformly distributed sample rates i.e. $T_n \sim$ U(0.01,0.02). The average sample rate in which case will be 0.015s.

The frequency response is then tabulated in Tables 3 and 4 accordingly. Figs. 6 and 7 show the obtained magnitude and phase plots, respectively, from the analysis of the three compensators, with frequencies between 0.1 to 10Hz.

Frequency (Hz)	Continuous (dB)	Uniform rate (dB)	Non-uniform rate (dB) max	min
0.1	19.98	19.8	19.7	19.7
0.5	19.6	19.62	19.62	19.62
1	18.6	18.67	18.67	18.67
2	16.3	16.32	16.39	16.31
3	14.4	14.34	14.4	14.2
4	13	12.89	12.95	12.6
5	12.1	11.87	12.27	11.5
6	11.4	11.16	11.45	10.6
7	10.9	10.65	11.8	9.4
8	10.7	10.1	11.7	8.9
9	10.5	11.16	11.7	8.1
10	10.4	10	12.4	7.4

Table 3. The filter magnitude values (dB) for the compensators under observation

Frequency (Hz)	Continuous (degrees)	Uniform rate (degrees)	Non-uniform rate (deg) max	min
0.1	-2.54	-2.8	-1.2	-1.2
0.5	-12.11	-13.4	-13.5	-13.4
1	-21.4	-24.25	-24.13	-24.35
2	-30.4	-35.7	-35.7	-36.15
3	-31.4	-39.2	-39.2	-40.3
4	-29	-38.6	-38.6	-40.7
5	-25.65	-38.8	-36.48	-42.3
6	-21.7	-39.25	-36.8	-42.4
7	-18.1	-36	-25.8	-45
8	-14.6	-35	-23	-47.2
9	-11.4	-34.5	-21.6	-47.2
10	-8.73	-34.55	-19	-48

Table 4. The filter phase values (degrees) for the compensators under observation

The error introduced by non-uniform sampling in the frequency response estimation is shown in Figs. 8 and 9. As compared to the case with uniform sampling, the results are similar at low frequencies. However, at higher frequencies the error keeps increasing depending on the amount of variations induced in the sample frequency.

5.1 Discussion

The simulations provide an illustration of a relatively simple system which is controlled by using the non-uniform rate control algorithm. The frequency response of the system is determined by using the Fourier analysis technique to estimate the frequency characteristics over time. The exercise makes use of a sampling regime with uniformly distributed sample rates to study the impact on the control performance. One apparent conclusion that is deduced from the simulations is that varying the distribution of the sampling rate does not provide any improvement in the phase lag of the system at low frequencies, since it remains the same when compared to the case with uniform sampling. In fact, as the operating frequency moves closer to the Nyquist limit, the variations in the sampling scheme become more apparent. This is largely due to the reconstruction process not having enough samples to reform the complete waveform.

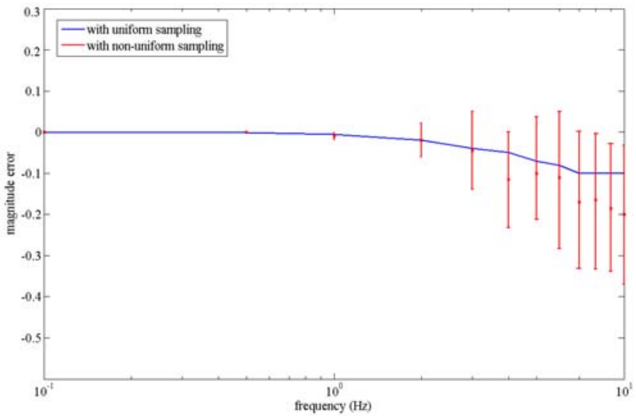

Fig. 8. The magnitude estimation error

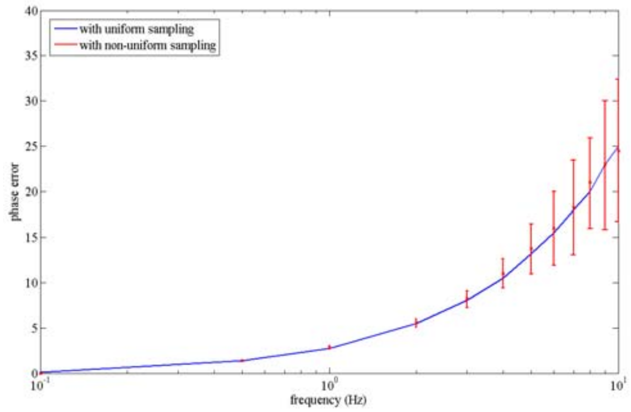

Fig. 9. The phase estimation error

6. CONCLUSION

This paper discusses the concept of non-uniform rate discrete equivalent controllers and highlights a technique to estimate the Bode plot using an approach based on the Fourier analysis. The technique is a combination of frequency domain and time domain analysis that plots the magnitude and phase of the signal in the time domain. After applying the technique on a practical compensator, it is concluded that, as far as the phase response is concerned, adopting a non-uniform sampling frequency does not provide any advantage over the uniform sample rate. However, the importance of the non-uniform rate control algorithm and its evaluation technique is a plausible development for control literature, which can be used to effectively process digital compensators with non-uniform sampling.

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